21-11-2011

n players

player i has a set of strategies S_i

strategy profile: $S = S_1 \times S_2 \times \dots \times S_n$

$$S = (\underset{\substack{\rotatebox{-90}{$\in$} \\ S_1}}{s_1}, \dots, \underset{\substack{\rotatebox{-90}{$\in$} \\ S_n}}{s_n})$$

mixed strategies of player i probability distribution on S_i

$$\Delta_i = \{ x \in \mathbb{R}_+^{S_i} \mid \sum_{s \in S_i} x(s) = 1 \} =$$

$$= \{ \sigma = (x_1, \dots, x_n) \in \mathbb{R}^{|S_i|} \mid \sum x_i = 1 \}$$

mixed strategy profiles $\Delta = \Delta_1 \times \Delta_2 \times \dots \times \Delta_n$

player's i utility $u_i : S \to \mathbb{R}$

Expected payoff: σ_i mixed profile

$$u_i(\sigma) = \mathbb{E}_{S \sim \sigma}[u_i(\sigma)] = \sum_{s \in S} \left(\prod_{i=1}^{n} \sigma_i(s_i) u_i(s) \right)$$

(Mixed) N.E. $\sigma = (\sigma_1, \dots, \sigma_n)$

$\forall_i, \forall \sigma_i' \in \Delta_i \quad u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$

equiv

① $\forall_i \ \forall s_i \in S_i \quad u_i(\sigma_i, \sigma_{-i}) \geq u_i(s_i, \sigma_{-i})$

ξεκάρουμε μόνο

για τις pure στρατηγικές

ότι δε βελτιώνεται τα κόστου

② $\forall i, \forall s_i, s_i' \in S_i$

$$\sigma_i(s_i) > 0 \Rightarrow u_i(s_i, \sigma_{-i}) \geq u_i(s_i', \sigma_{-i})$$

$\hookrightarrow s_i \in supp(\sigma_i)$

2 players 2 strategies

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 4 & 6 \end{bmatrix}$$

row player 1 $\qquad$ column player

	q	$1-q$
p	1,2	4,1
$1-p$	2,4	3,6

u_i = payoff of player 1 when he plays row i

v_j = payoff of player 2 when he plays row j

$$u_1 = q \cdot 1 + (1-q)4 = -3q + 4$$

$$u_2 = 2 \cdot q + 3(1-q) = -q + 3$$

$$v_1 = 2p + 4(1-p) = -2p + 4$$
$$v_2 = 1 \cdot p + 6(1-p) = -5p + 6$$

$$u_1 > u_2 \Rightarrow -3q + 4 > -q + 3 \Rightarrow q < 1/2$$

$$v_1 > v_2 \Rightarrow p > 2/3$$

Άρα $\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{2}{3}, \frac{1}{3}\right)$ is N.E.

καλό είναι να κοιτάω και τα pure N.E. εκτός από αυτά

LEMKE-HOWSON (for 2-players) 1964

2 players

$n \times m$ payoff matrices $A = (a_{ij})$, $B = (b_{ij})$

mixed strategies $x = (x_1, \dots, x_n) : \sum x_i = 1$

$y = (y_1, \dots, y_n) : \sum_{j=1}^{n} y_j = 1$

$$u_i = a_i \cdot y = \sum_{j=1}^{n} a_{ij} y_j$$

$$v_j = b_j^T \cdot x = \sum_{i=1}^{n} b_{ij} x_i$$

$$u = \max_i u_i, \quad v = \max_j v_j$$

$$x_i > 0 \Rightarrow u_i = u$$

$$\boxed{x_i (u - u_i) = 0} \quad (*)$$

↳ complementarity conditions

At N.E.

$$u_i = a_i y \quad v_j = b_j^T x$$

$u_i \leq u$

$v_i \leq v$

$x_i (u - u_i) = 0$

$y_i (v - v_i) = 0$

$1^T \cdot x = 1$

$1^T y$

$x_i, y_i \geq 0$

$X_i = u - u_i \quad , \quad Y_j = v - v_j$

divide by u, v

$$u - X_i = a_i Y$$
$$v - Y_j = b_j^T x$$
$$u - x_i \leq u$$
$$v - Y_j \leq v$$
$$x_i X_i = 0$$
$$Y_j Y_j = 0$$

$$1^T x = 1$$
$$1^T y = 1$$
$$x_i, y_i \geq 0$$

$$\frac{X_i}{u} = 1 - a_i \frac{Y}{u}$$
$$\frac{Y_j}{v} = 1 - b_j^T \frac{x}{v}$$
$$\frac{x_i}{u} \frac{X_i}{u} = 0$$
$$\frac{v_j}{v} \frac{Y_j}{u} = 0$$
$$\boxed{1^T \frac{x}{u} = \frac{1}{u}} \text{ DROP}$$
$$\left.\frac{x_i}{u} \frac{X_i}{u} \frac{Y_j}{u} \frac{Y_j}{u} \geq 0\right\}$$

MUST $u, v > 0$ → enough to assume $A, B > 0$

$\hat{X}_i = \frac{x_i}{u} \quad , \quad \hat{Y}_j = \frac{\hat{y}_j}{v}$

$\hat{x}_i = \frac{x_i}{v} \quad , \quad \hat{y}_j = \frac{y_j}{u}$

$$\hat{x}_i = 1 - a_i \hat{y}$$
$$\hat{y}_j = 1 - b_j^T \hat{x}$$

$$\left.\begin{array}{l} \hat{x}_i \hat{X}_i = 0 \\ \hat{y}_j \hat{Y}_j = 0 \end{array}\right\} \text{complementarities}$$

$$\hat{x}_i, \hat{y}_j, \hat{X}_i, \hat{Y}_j \geq 0$$

EXAMPLE

$$A = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 3 & 5 \end{bmatrix}$$

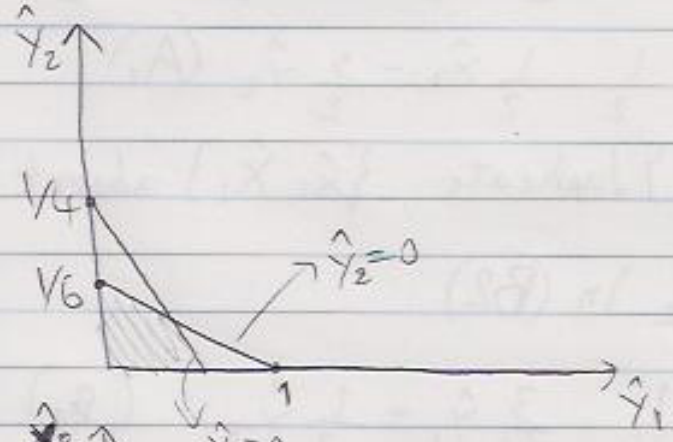
προσθέτω +1 σε όλα για να τα κάνω θετικά

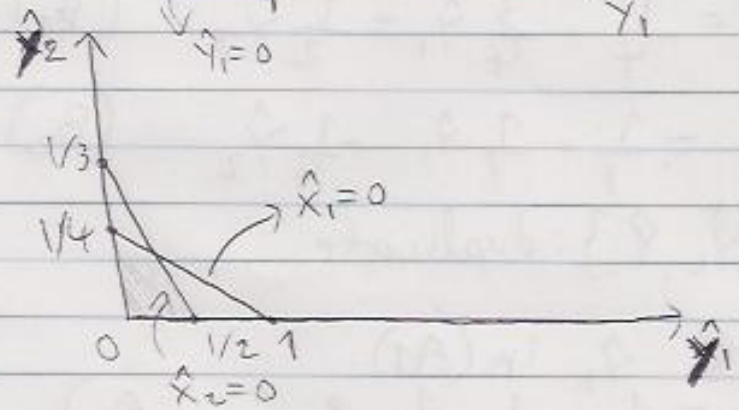
$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 4 & 6 \end{bmatrix}$$

$$(A) \begin{cases} \hat{x}_1 = 1 - \hat{y}_1 - 4\hat{y}_2 \\ \hat{x}_2 = 1 - 2\hat{y}_1 - 3\hat{y}_2 \end{cases}$$

← συνήθως έχουν αρνητικό συντελεστή οι λύσεις για να είναι κυρτό το πολύεδρο

$$(B) = \begin{cases} \hat{y}_1 = 1 - 2\hat{x}_1 - 4\hat{x}_2 \\ \hat{y}_2 = 1 - \hat{x}_1 - 6\hat{x}_2 \end{cases}$$





① Pivot $\hat{x}_1$ in $(B_1) \longrightarrow$ min-ratio-rule:
$(B_1): 1/2 \quad (B_2): 1$

(B) $\hat{x}_1 = \frac{1}{2} - \frac{1}{2}\hat{y}_1 - 2\hat{x}_2$

$\hat{y}_2 = 1 - \left(\frac{1}{2} - \frac{1}{2}\hat{y}_1 - 2\hat{x}_2\right) - 6\hat{x}_2$

$= \frac{1}{2} + \frac{1}{2}\hat{y}_1 - 4\hat{x}_2$

$\{\hat{x}_1, \hat{X}_1\}$ is not satisfied $\{\hat{y}_1, \hat{Y}_1\}$ duplicate

② pivot $\hat{y}_1$ in (A_2) min-ratio-rule

$$(A)\begin{cases} \hat{x}_1 = 1 - \frac{1}{2} - \frac{1}{2}\hat{x}_2 - \frac{3}{2}\hat{y}_2 = \frac{1}{2} + \frac{1}{2}\hat{x}_2 - \frac{5}{2}\hat{y}_2 & (A1) \\ \hat{y}_1 = \frac{1}{2} - \frac{1}{2}\hat{x}_2 - \frac{3}{2}\hat{y}_2 & (A_2) \end{cases}$$

$\{x_2, X_2\}$ duplicate, $\{\hat{x}_1, \hat{X}_1\}$ absent

③ pivot $\hat{x}_2$ in $(B2)$:

$$(B)\begin{cases} \hat{x}_1 = \frac{1}{4} - \frac{3}{4}\hat{y}_1 + \frac{1}{2}\hat{y}_2 & (B_1) \\ \hat{x}_2 = \frac{1}{8} - \frac{1}{8}\hat{y}_1 - \frac{1}{4}\hat{y}_2 & (B_2) \end{cases}$$

④ $\{\hat{y}_i, \hat{Y}_i\}$: duplicate

Pivot $\hat{y}_2$ in $(A1)$

$$(A)\begin{cases} \hat{y}_2 = \frac{1}{5} + \frac{1}{5}\hat{x}_2 - \frac{2}{5}\hat{x}_1 & (A_1) \\ \hat{y}_1 = \ldots = \frac{1}{5} - \frac{4}{5}\hat{x}_2 + \frac{3}{5}\hat{x}_1 & (A_2) \end{cases}$$

$\{\hat{x}_1, \hat{X}_1\}$

5 (0?!

By setting right hand side variables equal to 0 i.e $(\hat{x}_1 = \hat{x}_2 = \hat{y}_1 = \hat{y}_2 = 0)$ we get

$(A1)$: $\hat{y}_2 = \frac{1}{5}$, (A_2) $\hat{y}_1 = \frac{1}{5}$

$(B1)$: $\hat{x}_1 = \frac{1}{4}$, (B_2): $\hat{x}_2 = \frac{1}{8}$

Normalization

$$x_1 = \frac{\hat{x}_1}{\hat{x}_1 + \hat{x}_2} = \frac{1/4}{1/4 + 1/8} = \frac{2}{3}$$

$$x_2 = \frac{\hat{x}_2}{\hat{x}_1 + x_2} = \frac{1/8}{1/4 + 1/8} = 1/3$$

$$y_1 = \frac{\hat{y}_1}{\hat{y}_1 + \hat{y}_2} = \frac{1/5}{1/5 + 1/5} = 1/2$$

$$y_2 = \dots = 1/2$$